Midterm Exam. First year Electrical Engineering Dep.(Power 2017-2018)

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Answer the following question:

(1) If $w = \tan^{-1}(x^2 + y^2)$ Find $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}$.

(2) Solve the following differential equations $(D^2 - 5D + 6)y = e^x \sin 2x$

(3) Find the general solution of the differential equations
$xy'' - (2x+1)y' + (x+1)y = x^2 e^x$ given $y = e^x$ is a solution of homogenous equation

(4) Find the function y which satisfy the equation $x^2 y'' - 4xy' + (4x^2 + 6)y = x^4 \sin 2x$

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Answer

(1) If $w = \tan^{-1}(x^2 + y^2)$ Find $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}$

since $w = \tan^{-1}(x^2 + y^2)$ then $\tan w = (x^2 + y^2)$

$$\sec^2 w \frac{\partial w}{\partial x} = 2x \quad \Rightarrow \frac{\partial w}{\partial x} = 2x \cos^2 w \quad \Rightarrow x\frac{\partial w}{\partial x} = 2x^2 \cos^2 w$$

$$\sec^2 w \frac{\partial w}{\partial y} = 2y \quad \Rightarrow \frac{\partial w}{\partial y} = 2y \cos^2 w \quad \Rightarrow y\frac{\partial w}{\partial x} = 2y^2 \cos^2 w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 2x^2\cos^2 w + 2y^2 \cos^2 w = 2(x^2 + y^2)\cos^2 w = 2\tan w \cos^2 w$$

$$= 2\sin w \cos w = \sin 2w$$

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(2) $(D^2 - 5D + 6)y = e^x \sin 2x$

To find the complementary function construct the characteristic equation

$(m^2 - 5m + 6)y = 0$ Factorize to find the roots

$(m-3)(m-2) = 0$ The roots are $m = 3$ *and* $m = 2$

$y_c = Ae^{3x} + Be^{2x}$ Where A and B are arbitrary constants.

Now we find the particular integral

$$y_p = \frac{1}{(D^2 - 5D + 6)} e^x \sin 2x = \frac{1}{(D-2)(D-3)} e^x \sin 2x \text{ (use shift rule)}$$

$$= e^x \frac{1}{(D-1)(D-2)} \sin 2x = e^x \frac{1}{(D^2 - 3D + 2)} \sin 2x$$

$$= e^x \frac{1}{(-4 - 3D + 2)} \sin 2x = e^x \frac{1}{(-3D - 2)} \sin 2x$$

$$= -e^x \frac{(3D-2)}{(9D^2 - 4)} \sin 2x = -e^x \frac{(3D-2)}{(-36-4)} \sin 2x$$

$$= \frac{e^x}{40}(3D - 2) \sin 2x = \frac{e^x}{40}(6\cos 2x - 2\sin 2x)$$

The general solution $\boxed{y_G = y_c + y_p = Ae^{3x} + Be^{2x} + \frac{e^x}{40}(6\cos 2x - 2\sin 2x)}$

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(3) Find the general solution of the differential equations

$$xy'' - (2x+1)y' + (x+1)y = x^2 e^x \text{ given } y = e^x \text{ is a solution of homogenous equation}$$

Answer

$\boldsymbol{y = e^x}$ is a solution for the homogenous equation. Let the complementary function is $\boldsymbol{y = v\, e^x}$ then $y' = v'e^x + ve^x$ and $y'' = v''e^x + 2v'e^x + ve^x$

Substitute in the homogeneous equation

$$xy'' - (2x+1)y' + (x+1)y = x^2 e^x$$

$$v''xe^x + 2v'xe^x + vxe^x - (2x+1)(v'e^x + ve^x) + (x+1)ve^x = x^2 e^x$$

$v''x - v' = x^2$ put $u = v'$

$u'\mathrm{x} - \mathrm{u} = \mathrm{x}^2$ linear with integrating factor $\mu = \frac{1}{x}$

$\frac{1}{x}u' - \frac{1}{x^2}u = 1$ then $\frac{1}{x}u = x + c_1 \Rightarrow u = x^2 + c_1 x$

$v' = x^2 + c_1 x \quad \Rightarrow v = \frac{1}{3}x^3 + c_1 \frac{1}{2}x^2 + c_2$

$$\boxed{\boldsymbol{y_G = y = \tfrac{1}{3}x^3 e^x + \tfrac{1}{2}c_1 x^2 e^x + c_2 e^x}}$$

Another method

Factorize the equation

$$\left[xD-(x+1)\right]\left[D-1\right]y = x^2e^x$$

Put $\quad (D-1)y = u \qquad\qquad (1)$

$$\left[xD-(x+1)\right]u = x^2e^x \qquad\qquad (2)$$

Solve (2) and substitute in (1)

$$\left[xD-(x+1)\right]u = x^2e^x$$

$$xu'-(x+1)u = x^2e^x$$

$$u'-\frac{(x+1)}{x}u = xe^x$$

Liner differential equation with integral factor $\mu = e^{\int -\frac{(x+1)}{x}\mathrm{dx}} = e^{-x-\ln x} = \dfrac{e^{-x}}{x}$

Multiply the equation by $\dfrac{e^{-x}}{x}$

$$u'\frac{e^{-x}}{x}-\frac{(x+1)e^{-x}}{x^2}u = 1 \quad \therefore \frac{e^{-x}}{x}u = x+c_1 \ \ and \quad u = x^2e^x+c_1xe^x$$

Substitute in (1)

$$(D-1)y = x^2e^x+c_1xe^x$$

$$y'-y = x^2e^x+c_1xe^x$$

$$y'e^{-x}-ye^{-x} = x^2+c_1x$$

$$ye^{-x} = \tfrac{1}{3}x^3+\tfrac{1}{2}c_1x^2+c_2$$

$$\boxed{y = \tfrac{1}{3}x^3e^x+\tfrac{1}{2}c_1x^2e^x+c_2e^x}$$

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(4) $\boldsymbol{x^2y''-4xy'+(4x^2+6)y = x^4\sin 2x}$

Solution:

$$x^2y''-4xy'+(4x^2+6x)y = x^4\sin 2x$$

Write the equation in slandered form

$$y'' - \frac{4}{x}y' + (4 + \frac{6}{x})y = x^2 \sin 2x \quad p = -\frac{4}{x} \quad and \quad \frac{-1}{2}\int p(x)dx = \frac{-1}{2}\int \frac{-4}{x}dx = \ln x^2$$

Remove the first derivative by the transformation $y = uv$

where $v = e^{\frac{-1}{2}\int Pdx} = e^{\ln x^2} = x^2$

$$y = ux^2$$
$$y' = u'x^2 + 2xu$$
$$y'' = u''x^2 + 4u'x + 2u$$

Substitute in the equation $y'' - \frac{4}{x}y' + (4 + \frac{6}{x^2})y = x^2 \sin 2x$

$$u''x^2 + 4u'x + 2u - \frac{4}{x}(u'x^2 + 2xu) + (4 + \frac{6}{x^2})ux^2 = x^2 \sin 2x$$

$$u''x^2 + 4u'x + 2u - 4u'x - 8u + 4ux^2 + 6u = x^2 \sin 2x$$

$$u''x^2 + 4u'x + 2u - 4u'x - 8u + 4ux^2 + 6u = x^2 \sin 2x$$

$$u''x^2 + 4u'x + 2u - 4u'x - 8u + 4ux^2 + 6u = x^2 \sin 2x$$

$$u''x^2 + 4ux^2 = x^2 \sin 2x$$
$$u'' + 4u = \sin 2x$$
$$u_c = A\cos 2x + B\sin 2x$$
$$u_p = \frac{1}{D+4}\sin 2x = \frac{-x\cos 2x}{4}$$
$$u = A\cos 2x + B\sin 2x - \frac{x\cos 2x}{4}$$

The solution of the equation is $\boxed{y = uv = Ax^2\cos 2x + Bx^2\sin 2x - \frac{x^3\cos 2x}{4}}$